THE ACTION UNIT AS A PRIMARY UNIT IN THE SI

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Abstract

Action has been recognized as the most fundamental invariant of physics, since not only all classical dynamics can be derived from Hamilton's principle of stationary action, but also the Schrödinger equation of quantum mechanics and, more generally, any laws governing physical transformations, which may be described by a given number of state functions and of their first derivatives with respect to a given set of independent variables (generally space-time coordinates).

In addition, Heisenberg's uncertainty principle, which sets a fundamental limit to the measurement accuracy of any pair of conjugate quantities, can be best expressed by referring to action, as it implies that "action is quantized" and that therefore "the number of bits obtainable from a measurement of action cannot exceed the number of action quanta", as shown by the author.

From all the above, it stems that action is among the most fundamental quantities in physics (and, particularly, more fundamental than either energy or mass) and is directly tied with probability, together with the other fundamental quantity: information.

In fact, it is shown that action and information may be thought of as representing, respectively, the real and the imaginary part of the argument of the wave function in quantum mechanics.

In conclusion, action, in the opinion of the author, deserves to be assigned a specific measurement unit, also because it plays, in physics, the same unifying role, as that recognized by Giorgi in the joule (or the watt), towards the establishment of a coherent, rational system of units.

Note - The "SI" -- also called "MKS" or "MKSA" -- is the International System of measurement units, devised and proposed by Giovanni Giorgi on or about 1901 and bearing his name since about 1935, when it began to be internationally accepted.

1. Action in classical and quantum mechanics: a concise review

I hope that you will forgive a non-metrologist for daring to forward a proposal concerning the assignment of a specific measurement unit to "action", a physical quantity which is now about 2 $\frac{1}{2}$ centuries old, since its first introduction by Maupertuis in the year 1744; it is even more than 100 years older than the concept of energy, of which the scientific community was barely aware before the year 1860.

The purpose of my contribution is both to briefly review some known properties of action and to show some new ones, which add to the already rich list of important merits of this quantity.

It is first worth noting that the two well known principles of least action, the most ancient one, established by Pierre Louis Moreau de Maupertuis in 1744, and the one enunciated by Sir William Rowan Hamilton about 100 years later, are nothing but two different formulations of a general principle of economy in Nature, whereby a resource (i.e. some quantity, such as energy or momentum, which is not unlimitedly available in nature and therefore has to be conserved) is parsimoniously utilized to achieve change in space-time.

This is mathematically stated (fig. 1) by letting a certain variational integral *J* be stationary with respect to all possible paths that can be followed to achieve change from certain initial conditions to certain final conditions.

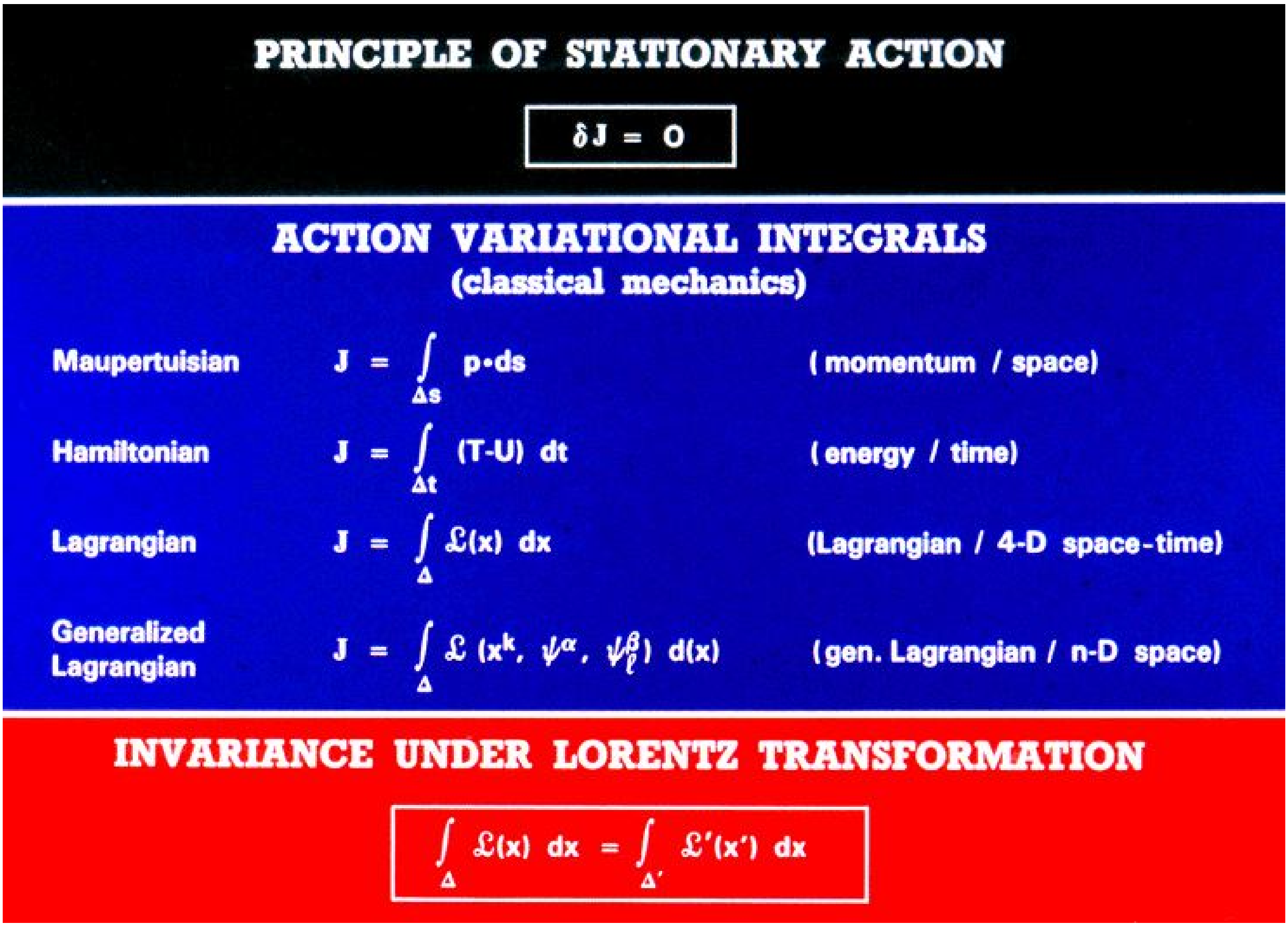

Fig. 1 - Relevant formulae concerning action variational integrals.

As is known, Maupertuisian formalism was concerned with the integral of momentum *p* along space *s*, whereas Hamiltonian formalism was concerned with the integral of energy (more precisely of the difference between kinetic (*T*) and potential (*U*)energy) along time *t*. Lagrangian formalism relates to the four-dimensional space-time and can be generalized to n-th-dimensional space. It is also pointed out that *J* is invariant under a Lorentz transformation.

At the beginning of this century, following the developments of the relativity theory, it was proposed by several authors [1] to consider time as the fourth coordinate of a four-dimensional space and to express action as the integral of the so-called Lagrangian density function (or "Lagrangian" for short) over a region Δ of the four-dimensional space-time domain. Using Lagrangian formalism, instead of Hamiltonian formalism, it is easily shown that action is invariant under a Lorentz transformation, i.e. that action is a scalar describing change, which has the same value for all inertial observers, no matter how high is their relative speed of motion or (which is the same) their speed relative to the observed object.

A further generalization of action has been achieved [2] by considering any physical system, the state of which can be described by *n* "independent" variables x^k (the values of which may be obtained from specific measurements) and by equations that correlate the *m* "dependent" variables ψ^α with the *n* independent variables x^k, subject to whatever boundary conditions that may be imposed on the transformation process under consideration.

The ψ^α are called the "state functions" of the system and, in a way, they represent our "a priori knowledge" about the transformation to be observed, together with the ψ_l^β, which are the first derivatives of a generic state function ψ^β with respect to a generic independent variable x^l. Of course, the volume element d(x) and the region of integration Δ refer to the n-th dimensional space established by the *n* independent variables x^k.

It may also be recalled that, in quantum mechanics, Lagrangian formalism is applied through the use of Hermitian operators, thus extending action invariance to the domain of microcosm, in addition to that outlined above, which is valid for the macrocosm [1, 3].

The power of the action variational integral is proved by the fact that all the basic conservation laws of physics can be derived -- as theorems -- from the principle of stationary action. This is the case for the conservation of linear and angular momentum, the conservation of energy and the conservation of center of mass in classical mechanics, as well as the conservation of the same quantities in field theories of quantum mechanics [2].

In addition, the equations of motion in classical mechanics, as well as the wave equations of quantum mechanics, can be derived from the stationary property of action, through a general and straightforward approach. In particular, Newton's three laws and the Schrödinger equation become theorems, derived from that single powerful principle of Physics which is the principle of stationary action.

2. Action, as seen by a "Boolean observer"

Let us now come to some new aspects concerning action, which have been suggested to the author from the widespread use of digital measuring instruments. These new achievements have somewhat changed, and perhaps clarified, the process of observation, which is basic to our perception of change and, more generally, to the acquisition of knowledge by an observer about the outer world.

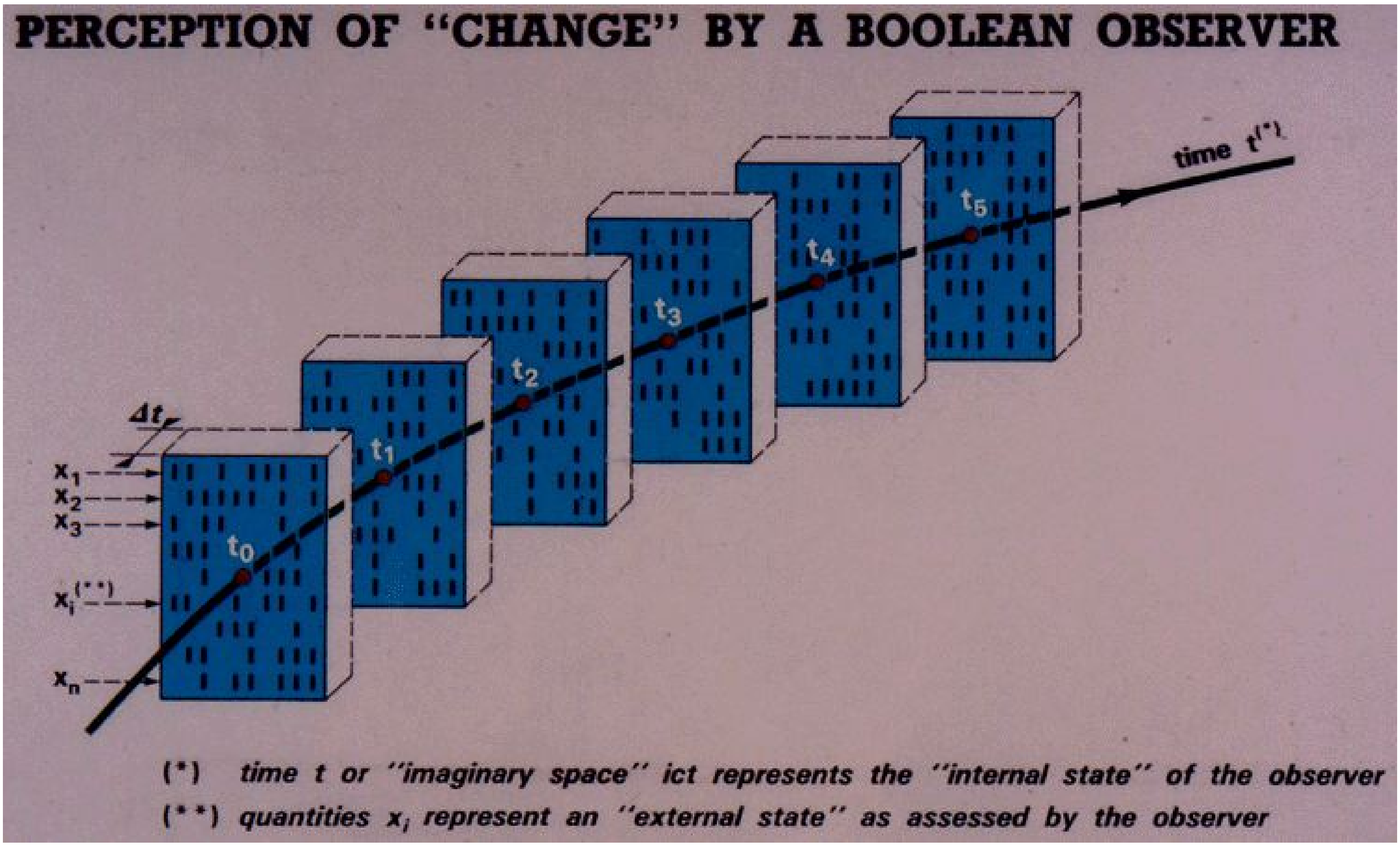

Fig. 2 - Perception of change by a "Boolean observer"

A digital measuring instrument, which we may call a "Boolean observer" [3], is one whose state of knowledge can be represented in terms of binary digits or bits. We may schematize the process of observation of a Boolean observer by a series of "state-cards" (fig. 2), in each of which a series of digital words (i.e. numbers), obtained from individual measurements of certain quantities x_i, are stored, in succession. This term "in succession" means that the state of affairs, that is external to the observer, is compared with events that are internal to the observer, to see how these external facts may be of relevance to the observer's internal facts. To this end, the observer has either a built-in master clock to open and close a number of observation gates or simply a measure of how time is progressing from birth to death. In other words, perception of change that is external to the observer is always performed against a perception of change that is internal to the observer and this latter is tied with that quantity which we call "time" (more precisely "proper time").

However, the "ordinality" of observations may not necessarily be performed with respect to time, i.e. with respect to internal changes of the observer (which criterion coincides with the Hamiltonian standpoint); it can be performed, from a more general standpoint, with respect to any other *observable*, external or internal to the observer (which is the Lagrangian standpoint). In other words, the state-cards of fig. 2 can be ordered with different criteria, e.g. according to increasing values of space coordinates instead of time coordinates, or to energy levels, or to speed or momentum values, or anything else.

In addition to "state cards", to be obtained from observations, a Boolean observer has a memory area in which all of its "a-priori knowledge" is stored: this is made up of equations, laws, fundamental constants, measurement units and "names" of quantities to be observed. Moreover, the observer allows for suitably sized, blank memory areas, to be destined for measurement results (x_i), according to their desired measurement accuracy.

One important remark concerning observations is that observation is quantized. This means that any Boolean observer cannot obtain a continuous knowledge of the outer world, for two basic reasons:

> i) because any unit of observation consumes the "time" of the observer (Δt in the figure), i.e. the observer's "lifetime;"
>
> ii) because any observation requires a perception of change and, therefore, a perception of an observation must be interleaved with a perception of "non-observation," i.e., it requires an interruption of the observation.

Either philosophers or psychologists may tell us that all the above is an obvious consequence of the fact that we generally base the acquisition of knowledge on the game of opposites, i.e. on binary states (ON-OFF) or, if you wish, on the "principle of two," which is that according to which our brain's left hemisphere works. An example of how our right hemisphere works is that of how we may sense a summer sunset, by annulling our identity and feeling that we are part of a whole, i.e. by getting a synthetic -- instead of an analytic -- knowledge of the environment, hence based on the "principle of one", as opposed to the "principle of two", which is the principle of Boolean logic and digital machines.

The above considerations about the invariance of action from the point of view of observers, i.e. of its capability of describing change from an absolute standpoint, suggest that action must have some direct relationship with the information obtained from observations, because information, too, is a description of change.

We believe that this relationship is intrinsically embedded in Heisenberg's uncertainty principle.

3. The relevance of action and information to the observer

The most usual formulations of the *uncertainty principle* deal with fundamental limitations imposed to the average accuracy with which measurements of two "conjugate" quantities can be performed "simultaneously". Energy and time, or momentum and position are frequently quoted as typical pairs of such quantities, strictly related to the space-time domain.

However, Heisenberg's uncertainty principle can also be, more generally, stated in terms of action, which, as is known, is dimensionally given by the product of either energy and time or momentum and position, as well as of a number of pairs of other conjugate quantities.

As the uncertainty principle applies to repeated measurements (made with the same equipment of the same type of event), we may state that, when taking many measurements of an action A involved in an experiment of the above type, we must expect that their "average" accuracy cannot be higher than $\hbar$ ($\hbar$ equaling the known Planck's constant h divided by 2π). As this statement also applies for $A = 0$, we may also state that:

"Any event, involving (on average) an action A less than $\hbar$, cannot be observed".

If, now, the observer is a Boolean observer (whereby bistable devices are its unique means of obtaining information from the environment), "no-knowledge obtained" means that the incoming stimulus (a photon or whatever other "messenger") is not able to fire even one single bistable device (e.g. a flip-flop circuit) of the observer, therefore it will produce no change in the observer's physical (hence informational) state. Thus, for a Boolean observer, Heisenberg's uncertainty principle can be also stated as follows:

"One cannot even obtain 1 bit of information from an event with $A < \hbar$".

At this point we may wish to turn the uncertainty principle into an affirmative form, i.e. to formulate a *certainty principle*, as follows:

" If $A = \hbar$, it is possible to obtain 1 bit of information "

and, of course,

" If $A = \alpha\,\hbar$ (with α an integer), it is possible to obtain α bits of information "

or, conversely:

" To obtain $\boldsymbol{I}$ bits of information, a minimum action $A = \hbar \boldsymbol{I}$ is required".

In conclusion, in the ideal case of maximum (Boolean) observation efficiency, the following relationship holds:

$$A = \hbar \boldsymbol{I} \qquad \textbf{(1)}$$

or, taking into account that $\alpha = A / \hbar$ represents the number of action quanta, we also obtain:

$$\alpha = \boldsymbol{I} \qquad \textbf{(2)}$$

i.e., in the ideal case,

" the number of action quanta equals the number of information bits and vice versa."

This is a more general statement than either "action is quantized" or "information is quantized", in that (fig. 3) it establishes that the number of (real) bricks, with which action is built-up, equals the number of (imaginary) bricks with which information -- hence the knowledge of an ideal (100% efficient) Boolean observer -- is built-up.

The kinship of action with information -- which appear as the two fundamental quantities involved with observation and change -- may also be confirmed through tracing them to their common root: "probability". We may, in fact, combine (fig. 4) Shannon's formula [4], defining information as a logarithmic measure of probability, with Feynman's demonstration [5] that action is nothing but the phase of the wave function $\overline{\psi}$ in quantum mechanics. We, thus, obtain an exponential expression of the wave function $\overline{\psi}$ where the complex argument of the exponential, which we may term "complex action", is given by a real part, representing the Hamiltonian action, and an imaginary part, which is proportional to binary information $\boldsymbol{I}$ (I_e representing the *neperian information).* [3]

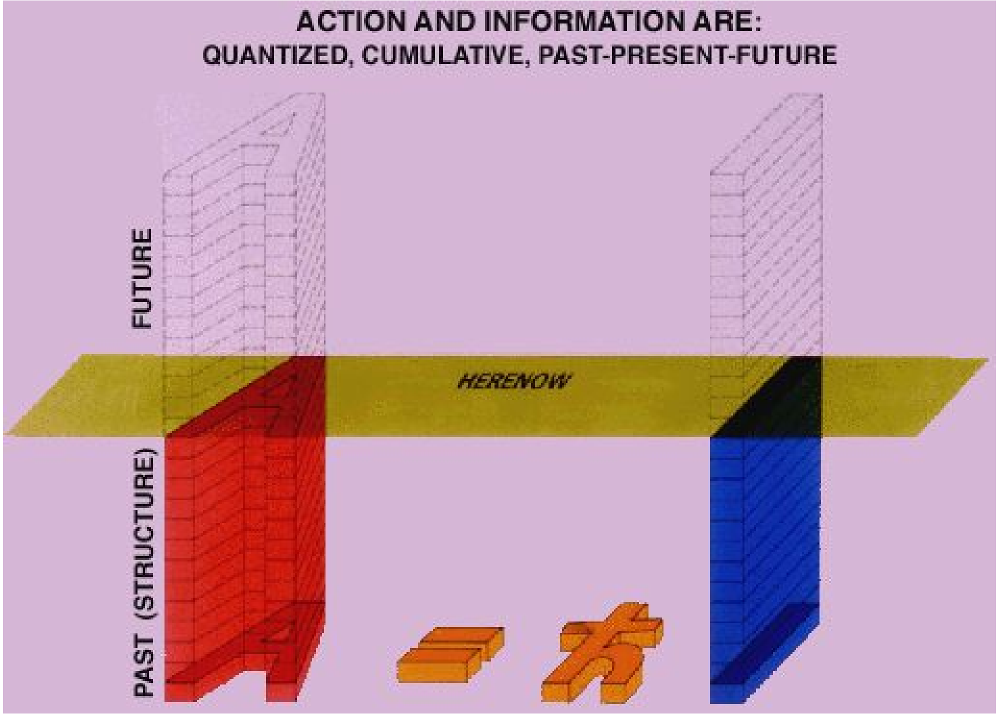

Fig. 3 - How action quanta and information bits add up, for an ideal Boolean observer.

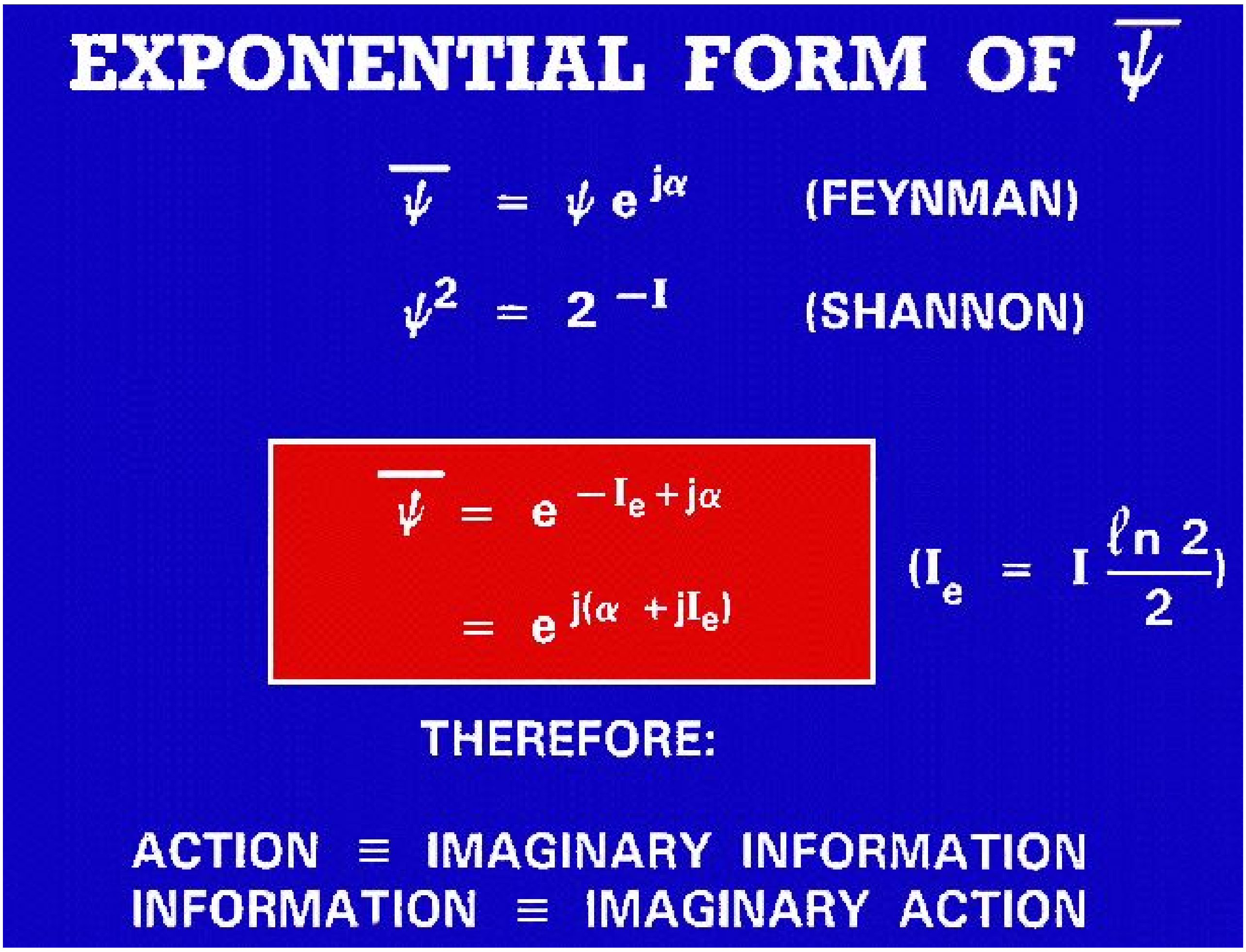

Fig. 4 - Combining information (Shannon) and action (Feynman) into a complex argument of the wave function.

Again, either a philosopher or a psychologist may tell us that this means that change is the combined result of physical forces amid of psychical forces (or the force of mind). However, more practically, this may simply confirm that the interaction between the subject and the object of an observation depends on both, as extensively proved by quantum mechanics.

Before closing this paragraph, the author wishes to acknowledge the pioneering work of Louis de Broglie (1948), who, following a first vague intuition by Sir Arthur Stanley Eddington, arrived at an equation

(equation (8), p. 89 of [6]), rewritten here:

$$A^0 / h = S / k \qquad (3)$$

where A^0 represents the "cyclic Maupertuisian action", S the "thermodynamic entropy", h the Planck constant and k the Boltzmann constant. It may be remarked that the quantity S/k represents the so-called "informational entropy", which is homogeneous to "Neperian information" (we recall that "cyclic action" implies that initial and final states coincide).

The similarity of (3) to (1) is evident, taking into account that A in (1) is a "progressive action" and A^0 in (3) is a "cyclic action."

4. Conclusions

In conclusion, we have shown that action plays a unique fundamental role in physics, being even conceivable as a quantity uprooted from space-time. Hence, action is the fundamental invariant describing any kind of change in the outer world of the observer, much the same as information is the fundamental invariant describing change in the inner world (mind) of the observer.

Other glorious quantities, such as energy or momentum, may be thought of as quantities derived from action, as they can be defined as follows:

- *energy* as the number of action quanta per unit time,
- *momentum* as the number of action quanta per unit length.

They may be thought of as means to describe how action, the primary cause of change, is seen by entities involved in such change, i.e. those who observe change and those whose change is observed.

It therefore appears that action may be seen, today, as the unifying quantity of all physical quantities, much the same as Giovanni Giorgi saw *energy* playing that role, in 1901.

As for the name to be given to the action measurement unit, the author believes that the choice is restricted to two names: *Maupertuis*, the first who introduced action, and *Planck*, the first who introduced the universal constant h.

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